Jour. Ind. Soc. Ag. Statistics
Vol. XXXIX, No. 3 (1987), pp. 271-279

INCOMPLETE BLOCK DESIGNS THROUGH ASYMMETRICAL FACTORIALS

R. C. JAIN and M. N. DAS*
I.A.S.R.I., New Delhi-12

(Received : February, 1985)

SUMMARY

Incomplete block designs have been obtained by using asymmetrical factorials both complete and fractional. The method consists of using different sets of numbers (not all equal) as the level codes for different factors. The incomplete block designs so obtained have flexible number of replications of different treatments. Some new series of designs have been obtained. Analysis of these designs has also been presented.

Keywords : Fractional factorial, Incomplete block design, Confounding.

Introduction

Designs for factorial experiments and Incomplete block designs are two main branches in the field of Design of experiments. Usually these are treated as separate topics as they serve different purposes. Construction of incomplete block designs was linked with factorial combinations for the first time by Yates [6] while constructing lattice designs which used to be called quasifactorials. Subsequently, Raghava Rao [5] pointed a link between orthogonal arrays and semiregular group-divisible partially balanced incomplete block designs. Jain and Das [4] gave a very simple algorithm for obtaining incomplete block designs through symmetrical factorials. The main result in this connection is that the treatment combinations in the factorial experiment can be utilised to provide blocks of incomplete block designs, i.e., each combination provides one block and

*Present address : I-1703, Chittaranjan Park, New Delhi-19.

the total number of combinations provide the entire incomplete block design.

In the present paper, incomplete block designs are obtained through asymmetrical factorial experiments. The incomplete block designs so obtained will have unequal number of replications. There are many occasions where designs with unequal number of replications are needed. Though some work has been done for obtaining designs with unequal number of replications, the present series of designs will provide quite a good number of designs needed by experimenters. These designs can also be analysed quite easily.

2. Method

Let there be k factors $A_1, A_2, \ldots, A_k$ arranged in some order. Let the number of levels of the ith factor A_i be p_i $(i = 1, 2, \ldots, k)$. The total number of treatment combinations from these levels of the factors is evidently $\pi\, p_i$. Let the level codes of the first factor A_1 be denoted by $1, 2, \ldots, p_1$; those for the second factor A_2 be $p_1 + 1, p_1 + 2, \ldots, p_1 + p_2$ and so on, the level codes for the last factor A_k being

$$1 + \sum_{i=1}^{k-1} p_i, \quad 2 + \sum_{i=1}^{k-1} p_i, \ldots, p_k + \sum_{i=1}^{k-1} p_i.$$

This type of designation of the level codes differs from the usual practice of using natural numbers starting from zero as level codes for each factor.

Next the factorial combinations are written using the above level codes and each of these treatment combinations is used as a block of an incomplete block design (IBD). This design has evidently, Σpi, treatments, number of blocks is equal to the total number of treatment combinations in the factorial, the block size is equal to the number of factors, k. The number of replications however differs from variety to variety. If the factorial is complete or is an orthogonal array of strength two or more, then the number of replications of each of the treatments designated by the level codes of the ith factor is equal to b/p_i $(i = 1, 2, \ldots, k)$. These incomplete block designs have thus unequal numbers of replications and can be analysed without much involvement as discussed in subsequent sections.

The method of construction discussed above is illustrated below by constructing an incomplete block design using the complete asymmetrical factorial $2 \times 3 \times 4$.

Let the three factors be denoted by A_1, A_2 and A_3 where A_1 has 2 levels, A_2 has 3 levels and A_3 has 4 levels. The level codes of A_1 are 1 and 2, those for A_2 are 3, 4 and 5 and those for A_3 are 6, 7, 8 and 9.

The treatment combinations written with these level codes are shown below.

1 3 6	2 3 6
1 3 7	2 3 7
1 3 8	2 3 8
1 3 9	2 3 9
1 4 6	2 4 6
1 4 7	2 4 7
1 4 8	2 4 8
1 4 9	2 4 9
1 5 6	2 5 6
1 5 7	2 5 7
1 5 8	2 5 8
1 5 9	2 5 9

Instead of viewing the above as 24 factorial combinations, we can consider the above as 24 blocks of an I.B.D., each combination providing a block. The IBD has the following parameters.

$$v = 9, \quad b = 24, \quad k = 3, \quad r = 12, 8 \text{ and } 6.$$

We have discussed in the next section the method of analysis of such incomplete block designs obtainable by using asymmetrical factorials which are either complete or are fractions of such factorials behaving as orthogonal arrays of strength two or more or Resolution 3 or more designs.

3. Analysis

Let the factorial design used for construction of an IBD be either complete or an orthogonal array of strength other than one, when the factorial is fractional. Let t_{ij} denote the effect of the treatment corresponding to the jth level of the ith factor ($i = 1, 2, \ldots k; j = 1, 2, \ldots, p_i$). Taking the usual model and applying the least squares method of estimation, the reduced normal equations of the design discussed above come out as below.

$$\left(r_i - \frac{r_i}{k}\right) t_{ij} - \frac{b}{kp_i} \sum_{l \neq i} \frac{S_l(t)}{p_l} = Q_{ij} \tag{1}$$

where $S_l(t)$ denotes the sum of all treatments corresponding to the levels of the lth factor, A_l. The above equation could be written conveniently by considering the fact that two treatments, one corresponding to the level of the ith factor and the other corresponding to the lth factor occur together in $b/p_i p_l$ blocks i.e. $\lambda_{il} = b/p_i p_l$ and two treatments corresponding to the levels of the same factor do not occur together in a block. Taking the restriction that

$$\sum_{l=1}^{k} \frac{S_l(t)}{p_l} = 0, \quad \text{the equation at (1) becomes}$$

$$\left(r_i - \frac{r_i}{k}\right) t_{ij} + \frac{b}{kp_i} \frac{S_i(t)}{p_i} = Q_{ij} \tag{2}$$

Adding equation at (2) over j, we get

$$\left(r_i - \frac{r_i}{k}\right) S_i(t) + \frac{b}{kp_i} S_i(t) = S_i(Q) \tag{3}$$

where $S_i(Q) = \sum_{j=1}^{p_i} Q_{ij}$

From equation (3), we get

$$S_i(t) = \frac{S_i(Q)}{r_i} \quad \text{where } r_i = \frac{b}{p_i}$$

Substituting this value of $S_i(t)$ in (2), we get

$$t_{ij} = \frac{k}{(k-1)\, r_i} Q_{ij} - \frac{1}{r_i p_i (k-1)} S_i(Q) \tag{4}$$

After the estimates of t_{ij} are obtained, then the adjusted treatment $S.S.$ is obtained as usual from $\Sigma\, t_{ij} Q_{ij}$.

Different types of variances of treatment contrasts are given below.

$$V(t_{ij} - t_{ij}') = \frac{2k}{(k-1)\, r_i} \sigma^2$$

$$V(t_{ij} - t_{mj}') = \frac{1}{(k-1)} \left[\frac{k}{r_i} + \frac{k}{r_m} - \frac{1}{r_i p_i} - \frac{1}{r_m p_m}\right] \sigma^2$$

4. Use of Factorials with Confounding of Some Two Factor Interactions

Certain types of fractional factorials where two factor interactions are confounded can also be used for obtaining incomplete block designs by following exactly the same method described earlier but its analysis is not the same as for the designs discussed above. Such designs are available with smaller number of blocks.

One method of obtaining fractions of factorials is to define an Identity group of interactions by following the method of Finney [3] and Das and

Giri [2]. If any two factor interaction is present in the Identity group of interactions then we shall say that in the fraction a two factor interaction is confounded. We shall use only that type of fraction where one two factor interaction is confounded.

We shall use the factor, treatment etc. notations as in the previous section. Let the affected interaction be denoted by $A_i\, A_m$. Let further the number of levels of A_i and A_m be each p so that $p_i = p_m = p$. The reduced normal equation for a treatment belonging to the levels of A_i or A_m come out as below :

$$\left(r_i - \frac{r_i}{k}\right) t_{ij} - \frac{b}{k} \sum_{l \neq m,\, i} S_l(t) - \frac{\lambda_{im}}{k}\, t_{mj} - \frac{\lambda_{im'}}{k} \sum_{j' \neq j} t_{mj'} = Q_{ij} \tag{5}$$

Where t_{mj} is a treatment coming from the level of mth factor such that it occur with t_{ij} in λ_{im} blocks. While t_{ij} and any other treatment $t_{mj'}$ do not either occur together in a block or occur together in $\lambda_{im'}$ blocks. The estimate of $S_l(t)$ remains the same as earlier $S_l(Q)/r_l$. Substituting for $S_l(t)$, the above equation reduces to

$$\left(r_i - \frac{r_i}{k}\right) t_{ij} - \frac{\lambda_{im}}{k}\, t_{mj} - \frac{\lambda_{im'}}{k} \sum_{j' \neq j} t_{mj'} = Q_{ij} + \frac{1}{kp} \sum_{l \neq i,\, m} S_l(Q) \tag{6}$$

Similarly equation for t_{mj} can be written. Taking $r_i = r_m = r$ and adding the two equations corresponding to t_{ij} and t_{mj} we get

$$\left(r - \frac{r}{k}\right)(t_{ij} + t_{mj}) - \frac{\lambda_{im}}{k}(t_{ij} + t_{mj}) - \frac{\lambda_{im'}}{k} \sum_{j' \neq j} (t_{mj'} + t_{ij'}) = Q_{ij} + Q_{mj} + \frac{2}{kp} \sum_{l \neq i,\, m} S_l(Q)$$

or

$$\left(r - \frac{r}{k} - \frac{\lambda_{im}}{k} + \frac{\lambda_{im'}}{k}\right)(t_{ij} + t_{mj}) = Q_{ij} + Q_{mj} + \frac{2}{kp} \sum_{l \neq i,\, m} S_l(Q) + \frac{\lambda_{im'}}{kr}\,[S_m(Q) + S_i(Q)] \tag{7}$$

Using equation (6) and the solution of $(t_{ij} + t_{mj})$ from (7), we get the estimate of t_{ij} as

$$t_{ij} = \frac{rk'}{(rk')^2 - \lambda_i^2} Q_{ij} + \frac{\lambda_i}{(rk')^2 - \lambda_i^2} Q_{mj} + \frac{1}{kp(rk' - \lambda_i)} \sum_{l \neq i,\, m} S_l(Q) + \frac{k' \lambda_{im}'}{k[(rk')^2 - \lambda_i^2)]} S_m(Q) + \frac{\lambda_i \lambda_{im}'}{kr[(rk')^2 - \lambda_i^2]} S_i(Q) \tag{8}$$

where

$$k' = \frac{k-1}{k} \quad \text{and} \quad \lambda_i = \left(\frac{\lambda_{im}}{k} - \frac{\lambda_{im'}}{k} \right)$$

The estimates for treatments which come from the levels of unaffected factors remain the same as for designs discussed in the previous section. After the solutions are obtained the *S.S.* can be obtained by usual methods. The variance of different types of treatment contrasts can also be obtained by the method of collection of coefficients of Q_{ij}. Such variances have been obtained in detail in the illustration.

5. Illustration

Consider $\frac{1}{3}$ (4×3^2). The fraction is obtained through the technique of Das [1]. Here only the principal block of one replication is needed. The three factors are denoted by X at four levels, A at three levels and B at three levels. In this case two factor interaction AB is affected. The fractional factorial and incomplete block design obtained through it are presented side by side.

Fractional factorial X A B	*Incomplete block designs*
0 0 0	0 4 7
0 1 2	0 5 9
0 2 1	0 6 8
1 0 2	1 4 9
1 2 0	1 6 7
1 1 1	1 5 8
2 1 0	2 5 7
2 0 1	2 4 8
2 2 2	2 6 9
3 2 0	3 6 7
3 0 2	3 4 9
3 1 1	3 5 8

The incomplete block design has the parameters $v = 10$, $b = 12$, $k = 3$, $r_1 = 3$, $r_2 = r_3 = 4$.

The normal equations for estimating the treatment effects on the usual model are shown below for typical cases.

$$(3 - \tfrac{3}{3})\, t_0 - \tfrac{1}{3}\,[S_2(t) + S_3(t)] = Q_0 \tag{9}$$

$$(4 - \tfrac{4}{3})\, t_4 - \tfrac{2}{3}\, t_9 - \tfrac{1}{3}\, t_7 - \tfrac{1}{3}\, t_8 - \tfrac{1}{3}\, S_1(t) = Q_4 \tag{10}$$

$$(4 - \tfrac{4}{3})\, t_9 - \tfrac{2}{3}\, t_4 - \tfrac{1}{3}\, t_5 - \tfrac{1}{3}\, t_6 - \tfrac{1}{3}\, S_1(t) = Q_9 \tag{11}$$

In equation (10) the ith and mth factors are the second and the third factor i.e. A and B respectively. If we take t_{ij} as treatment number 4, then t_{mj} the treatment corresponding to t_4 is t_9 because the other treatments from the levels of the factor B are t_7 and t_8 which are occurring with t_4 in equal number of blocks in the design.
Thus $\lambda_{im} = 2$ and $\lambda_{im'} = 1$.
The equation (10) can be rewritten as $(4 - \tfrac{4}{3})\, t_4 - \tfrac{1}{3}\, t_9 - \tfrac{1}{3}$

$$[S_1(t) + S_3(t)] = Q_4 \tag{12}$$

It can be shown by adding the normal equations for t_0, t_1, t_2 and t_3 and taking the restriction that $\Sigma\, S_i(t)/p_i = 0$, the estimate of $S_1(t) = S_1(Q)/r_1 = S_1(Q)/3$. Similarly by adding the normal equations for t_4, t_5 and t_6, we get the estimate of $S_2(t) = S_2(Q)/r_2 = S_2(Q)/4$. Similarly the estimate of $S_3(t) = S_3(Q)/4$. Substituting these estimates in equation (9), we get

$$t_0 = Q_0/2 - S_1(Q)/24 \text{ since } \sum_{i=1}^{3} S_i(Q) = 0$$

From equation (12), we get

$$(4 - \tfrac{4}{3})\, t_4 - \tfrac{1}{3}\, t_9 = Q_4 + \tfrac{1}{3}\,[S_1(Q)/3 + S_3(Q)/4]$$

Similarly the equation for t_9 is

$$(4 - \tfrac{4}{3})\, t_9 - \tfrac{1}{3}\, t_4 = Q_9 + \tfrac{1}{3}\,[S_1(Q)/3 + S_2(Q)/4]$$

Solving these two equations, we get

$$t_4 = 8Q_4/21 + Q_9/21 + S_1(Q)/21 + S_2(Q)/252 + 2\,S_3(Q)/63$$

$$t_9 = Q_4/21 + 8\,Q_9/21 + S_1(Q)/21 + 2\,S_2(Q)/63 + S_3(Q)/252$$

Using $\Sigma\, S_i(Q) = 0$, these reduce to

$$t_4 = 8Q_4/21 + Q_9/21 + S_1(Q)/63 - S_2(Q)/36 \text{ and}$$

$$t_9 = Q_4/21 + 8Q_9/21 + S_1(Q)/63 - S_3(Q)/36$$

These estimates are evidently abtainable from equation (8) in Section 4. In general the estimate for any one of the treatments t_4 to t_9 is given by

$$t_i = 8\, Q_i/21 + Q_j/21 + S_1(Q)/63 - S_i(Q)/36$$

where Q_j corresponds to the treatment t_j which occurs with t_i in λ_{im} blocks, that is, 2 blocks and $S_i(Q)$ is the sum of adjusted totals for that treatment which includes t_i, a treatment group being formed for the treatments from the levels of a factor.

The general solution for any of the treatments t_0, t_1, t_2 and t_3 is given by

$$t_u = Q_u/2 - S_u(Q)/24$$

where $S_u(Q)$ is the same as $S_1(Q)$. These estimates are evidently obtainable from equation (4) in Section 3.

Variances of different treatment contrasts are given below.

$$(V(t_u - t_{u'}) = \sigma^2,\ V(t_i - t_{i'}) = 16\,\sigma^2/21$$
$$V(t_i - t_j) = 11\,\sigma^2/18,\ V(t_u - t_i) = 401\,\sigma^2/508$$
$$V(t_i - t_{j'}) = 89\,\sigma^2/126.$$

6. Series of Designs

Some series of designs which can be obtained by using both complete and fractional asymmetrical factorials are discussed below.

6.1 *IBD Obtainable from Complete Asymmetrical Factorials*

When complete factorial ts used, we get IBD with parameters $v = \sum_{i=1}^{k} p_i,\ b = \pi\, p_i,\ k = k,\ r_i = b/p_i$ where k is the number of factors in asymmetrical factorial and p_i the levels of ith factor.

6.2 *IBD from Fractional Asymmetrical Factorials in which no Two Factor Interactions are Confounded*

If the asymmetrical factorial is of the form $p_1 \times p_2 \times \ldots \times p_n \times p^m$ where $m > 2$, then it is always possible to get a fraction of the above design where no two factor interactions are confounded. This fraction can be obtained by restricting confounding among the m factors each at p levels. If p^t is the fraction size of p^m so that no two factor interactions among the m factors are affected, then the total fraction size of the asymmetrical factorial will be $p_1 \times p_2 \times \ldots \times p_n \times p^t$. The IBD obtained from the above fraction will have the following parameters.

$$v = \sum_{i=1}^{n} pi + mp, b = p_1 \times p_2 \ldots \times p_n \times p^t, k = n + t, r = b/p_i, b/p,$$

number of factors $= n + m$ (n factors at pi levels and m factors each at p levels).

6.3 *IBD Obtainable from Fractional Asymmetrical Factorial in which only One Two Factor Interaction is Confounded*

A fraction of size $p_1 \times p$ of the asymmetrical factorial $p_1 \times p^2$ can be obtained by following the method of Das [1] in which only one two factor interaction will be affected. We can get these designs in blocks of size three. IBD obtainable from such fractions will have parameters.

$$v = p_1 + 2p, \quad b = p_1 p, \quad k = 3, r = \frac{b}{p_1}, \frac{b}{p}$$

A fraction of size p^2 can also be obtained by the method of Das [1] but in this case, two factor interactions alongwith a main effect will be affected. This will also give IBD but this has not been considered as we have not given any general technique of analysis of IBD obtainable through fractional factorial in which main effect and more than one two factor interaction are affected.

REFERENCES

[1] Das, M. N. (1960) : Fractional replicates as asymmetrical factorial designs, *Jour. Ind. Soc. Ag. Statistics*, **12** : 159-174.

[2] Das, M. N. and Giri, N. C. (1979). *Design and Analysis of Experiments*, Wiley Eastern Limited, New Delhi.

[3] Finney, D. J. (1945) : The fractional replication of factorial arrangements, *The annals of Eugenics*, **12** : 291-301.

[4] Jain, R. C. and Das, M. N. (1985): Incomplete block designs through symmetrical factorials, *Jour. Ind. Soc. Ag. Statistics*, **37** : 252-260.

[5] Raghavarao, D. (1971) : *Constructions and Combinatorial Problems in Design of Experiments*, Wiley, New York.

[6] Yates, F. (1936) : A new method of arranging variety trials invotving a large number of varieties, *Ann. Eugen.*, **7** : 319-22.